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LETTER TO THE EDITOR

The Clebsch–Gordan coefficients and isoscalar factors of the graded unitary group SU(m/n)

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Abstract. It is pointed out that the Clebsch–Gordan coefficients (CGC) of the permutation group are the coefficients which couple irreducible bases (IRB) of $SU(m/n)$ and $SU(p/q)$ to the IRB of $SU(mp+nq/mp+np)$, and the outer-product reduction coefficients (ORC) of the permutation group are the coefficients which couple the IRB of $SU(m/n)$ and $SU(p/q)$ into the IRB of $SU(m+p/n+q)$. The ORC of permutation groups with due modification in signs are the CGC for the special Gel'fand basis of $SU(m/n)$. The isoscalar factor (ISF) for the permutation group chain $S(f) \supset S(f_1) \times S(f_2)$ is the ISF for the graded unitary group chain $SU(mp+nq/mq+np) \supset SU(m/n) \times SU(p/q)$, and the outer-product ISF for the group chain $S(f) \supset S(f_1) \times S(f_2)$ is the ISF for $SU(mp+nq/mq+np) \supset SU(m/n) \otimes SU(p/q)$. All these coefficients can be calculated easily and some are already available.

The graded unitary group $SU(m/n)$, together with its possible applications in particle and nuclear physics, has been the subject of much research (Ne'eman 1979, Dondi and Jarvis 1979, 1981, Jarvis and Green 1979, Taylor 1979, Balantekin and Bars 1981a, b, Balantekin *et al* 1981, Sun and Xan 1981, Xan *et al* 1981). Dondi and Jarvis (1981) and Balantekin and Bars (1981a) independently introduced the graded permutation group and showed that the irreducible representation (IRR) of $SU(m/n)$ can be labelled by the graded Young diagram. Furthermore, Dondi and Jarvis (1981) have shown that for a large class of representations, the Young diagram techniques for Krönecker products, branching rules, dimension formulae, plethysyms and so on can be continued (with suitable modifications) into the graded case. However, to the best of our knowledge, the calculation of the CGC and ISF for the $SU(m/n)$ group remains an open question.

We have successively solved the problem of the CGC and ISF for the ordinary unitary group in terms of the ordinary permutation group (Chen *et al* 1978a, b, Chen 1981, Chen and Gao 1982b). The advantage of this permutation group approach to the CGC or ISF of the unitary group lies in the fact that the results obtained are independent of the rank of the unitary group being considered. By an extension from the permutation group to the graded permutation group, all the results related to the ordinary unitary group (Chen *et al* 1977b, Chen *et al* 1978a,b, Chen 1981) can be transferred (with slight modifications) to the graded unitary group $SU(m/n)$. In this letter we sketch some new, but surprisingly simple results concerning the CGC and ISF of $SU(m/n)$. Detailed accounts will be published elsewhere.

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Throughout the paper, we deal only with the so-called class I representations (Balantekin and Bars 1981a), i.e. the case with state indices $A = a = 1, 2, \ldots, m$ representing bosonic states and $A = \alpha = m+1, \ldots, m+n$ representing fermionic states. Most of our notation follows Chen (1981) and Chen and Gao (1982b). In the following we quote the results for the ordinary and graded cases in parallel to facilitate comparison.

1. The Casimir invariants

(*a*) Partensky (1972a, b) showed that, in the space of f particle product states, the kth power Casimir invariants I_k^m of the group U(m) are functions of the i-cycle class operators $C_{(i)}(f)$, $i = k, k-1, \ldots, 2$, of the permutation group S(f), as well as of the quantity m, namely

$$I_k^m = F_k(C_{(k)}(f), C_{(k-1)}(f), \ldots, C_{(2)}(f), m) \qquad k = m, m-1, \ldots, 2, 1 \tag{1a}$$

where F_k denote functional relations.

(*b*) It can be shown (Chen *et al* 1982a) that the Casimir invariants $I_k^{m/n}$ of U(m/n) can be simply obtained from equation (1a) by the substitutions

$$C_{(i)}(f) \to \mathring{C}_{(i)}(f) \qquad m \to m-n. \tag{2}$$

This gives

$$I_k^{m/n} = F_k(\mathring{C}_{(k)}(f), \mathring{C}_{(k-1)}(f), \ldots, \mathring{C}_{(2)}(f), m-n) \qquad k = m+n, m+n-1, \ldots, 2, 1 \tag{1b}$$

where $\mathring{C}_{(i)}(f)$ is the i-cycle class operator of the graded permutation group $\mathring{S}(f)$, which is isomorphic to the ordinary permutation group S(f).

Equation (*b*) is crucial for all the following results. It shows that the Casimir operators of U(m/n) are functions of the CSCO I (complete set of commuting operators of the first kind, the counterpart of the Casimir operators) of $\mathring{S}(f)$ (Chen *et al* 1977a, Chen and Gao 1982a). Therefore if a basis vector belongs to the IRR (ν) of SU(m/n), it must also belong to the IRR (ν) of $\mathring{S}(f)$ and *vice versa*. Consequently we can use partitions to label the IRR of SU(m/n) and S(f).

2. The Gel'fand basis of SU(m/n)

(a) The so-called quasi-standard (or quasi-Yamanouchi) basis of the state permutation group has been identified with the Gel'fand basis of SU(m) (Chen *et al* 1977b, Chen and Gao (1982a).

(b) Similarly, the quasi-standard basis of the graded state permutation group can be identified with the Gel'fand basis of the SU(m/n) group, i.e. the IRB classified according to the group chain SU(m/n) ⊃ SU($m/n-1$) ⊃ ... ⊃ SU(m) ⊃ SU($m-1$) ⊃ ... ⊃ SU(2) ⊃ U(1) (Chen *et al* 1982a). We can use a graded (or super) Weyl tableau $\mathring{W}_k^\nu$ to label a SU(m/n) Gel'fand basis, ν being the partition label and k the component indices. For example,

$$\begin{array}{|c|c|c|c|} \hline a & b & \alpha & \gamma \\ \hline b & \alpha \\ \cline{1-2} \beta \\ \cline{1-1} \end{array}$$

	SU(2/3) ⊃	SU(2/2) ⊃	SU(2/1) ⊃	SU(2) ⊃	U(1)
belongs to the IRR	[421]	[321]	[32]	[21]	[1]

where $a, b, c, \ldots$ and $\alpha, \beta, \gamma, \ldots$ are the state indices for bosons and fermions respectively.

3. The $\mathrm{SU}(mp+nq/mq+np) \supset \mathrm{SU}(m/n) \times \mathrm{SU}(p/q)$ IRB

(a) Suppose $|Y^{\nu_1}_{m_1}, W^{\nu_1}_1\rangle$ ($|Y^{\nu_2}_{m_2}, W^{\nu_2}_2\rangle$) is the Yamanouchi basis $[\nu_1]m_1$ ($[\nu_2]m_2$) of the permutation group S(f), as well as the IRB $[\nu_1]W_1$ ($[\nu_2]W_2$) of the group SU(m) (SU(n)) in the x (ξ) space, where $Y^{\nu_i}_{m_i}$ are the Young tableaux, m_i the Yamanouchi symbols, and $W^{\nu_i}_i \equiv W_i$ the Weyl tableaux. The $\mathrm{SU}(mn) \supset \mathrm{SU}(m) \times \mathrm{SU}(n)$ IRB can be constructed in terms of the CGC of the permutation group S(f) (Chen 1978b)

$$|Y^{\nu}_{m,\beta\nu_1 W_1 \nu_2 W_2}{}^{[\nu]}\rangle = \sum_{m_1 m_2} C^{[\nu]\beta,m}_{\nu_1 m_1, \nu_2 m_2} |Y^{\nu_1}_{m_1} W_1\rangle |Y^{\nu_2}_{m_2} W_2\rangle. \tag{3a}$$

Here the multiplicity label β distinguishes between repeated IRR of ($[\nu_1][\nu_2]$) in the IRR $[\nu]$ of SU(mn). Tables of the CGC for S(2)–S(6) are available either in the square root form of rationals (Chen and Gao 1981), or in decimal form (Shindler and Mirman 1977).

(b) The same CGC of the permutation group S(f) can be used to couple the IRB of SU(m/n) and SU(p/q) to the IRB of SU($mp+nq/mq+np$)

$$|\mathring{Y}^{\nu}_{m,\beta\nu_1 \mathring{W}_1 \nu_2 \mathring{W}_2}{}^{[\nu]}\rangle = \sum_{m_1 m_2} C^{[\omega]\beta,m}_{\nu_1 m_1, \nu_2 m_2} |\mathring{Y}^{\nu_1}_{m_1}, \mathring{W}_1\rangle |\mathring{Y}^{\nu_2}_{m_2}, \mathring{W}_2\rangle. \tag{3b}$$

Equation (3b) is the Yamanouchi basis $\mathring{Y}^{\nu}_{m}$ of the graded permutation group in the (x, ξ) space, as well as the $\mathrm{SU}(mp+nq/mq+np) \supset \mathrm{SU}(m/n) \times \mathrm{SU}(p/q)$ IRB, $\mathring{Y}^{\nu_i}_{m_i}$ and $\mathring{W}_i$ being the graded Young tableaux and graded Weyl tableaux respectively.

In the special case when all single-particle states are bosonic or fermionic, the graded Young and Weyl tableaux are simply related to the ordinary Young tableaux and Weyl tableaux by

$$|\mathring{Y}^{\nu}_{m} \mathring{W}^{\nu}_{k}\rangle = \begin{cases} |Y^{\nu}_{m} W^{\nu}_{k}\rangle & \text{for total bosonic} \quad (3c) \\ |\tilde{Y}^{\nu}_{m} \tilde{W}^{\nu}_{k}\rangle & \text{for total fermionic} \quad (3d) \end{cases}$$

where a tilde denotes conjugation tableaux (interchange of rows with columns).

4. The $\mathrm{SU}(m+p/n+q) \supset \mathrm{SU}(m/n) \otimes \mathrm{SU}(p/q)$ IRB

(a) Let the numbers $1, 2, \ldots, f$ be divided into two subgroups (ω_1) and (ω_2) consisting of f_1 and $f_2 = f - f_1$ numbers in ascending order

$$(\omega_1) = (a_1, a_2, \ldots, a_{f_1}) \qquad (\omega_2) = (a_{f_1+1}, \ldots, a_f).$$

The $\binom{f}{f_1}$ sequences $(\omega) = (\omega_1, \omega_2)$ are referred to as the normal order sequences.

The IRB $|Y^{\nu_1}_{m_1}(\omega_1), W_1\rangle$ of $\mathrm{S}(f_1) = \mathrm{S}(\omega_1)$ and SU(m), and those $|Y^{\nu_2}_{m_2}(\omega_2), W_2\rangle$ of $\mathrm{S}(f_2) = \mathrm{S}(\omega_2)$ and SU(n) can be coupled into the IRB of S(f) and SU($m+n$) by means of the ORC $C^{[\nu]\beta,m}_{\nu_1 m_1, \nu_2 m_2, \omega}$ (Chen *et al* 1978a, b)

$$|Y^{\nu}_{m,\beta\nu_1 W_1 \nu_2 W_2}{}^{[\nu]}\rangle = \sum_{m_1 m_2 \omega} C^{[\nu]\beta,m}_{\nu_1 m_1, \nu_2 m_2, \omega} |Y^{\nu_1}_{m_1}(\omega_1), W_1\rangle |Y^{\nu_2}_{m_2}(\omega_2), W_2\rangle. \tag{4a}$$

Tables of the ORC for S(2)–S(6) in the square root form of rationals along with the program in ALGOL 60 have been published (Chen and Gao 1981).

(b) The relation (4a) remains true for the graded case

$$|\mathring{Y}^{\nu}_{m,\beta\nu_1\mathring{W}_1\nu_2\mathring{W}_2}{}^{[\nu]}\rangle = \sum_{m_1m_2\omega} C^{[\nu]\beta,m}_{\nu_1m_1,\nu_2m_2,\omega}|\mathring{Y}^{\nu_1}_{m_1}(\omega_1), \mathring{W}_1\rangle|\mathring{Y}^{\gamma_2}_{m_2}(\omega_2), \mathring{W}_2\rangle \quad (4b)$$

where $|\mathring{Y}^{\nu_1}_{m_1}(\omega_1)\mathring{W}_1\rangle$ is the IRB of $\mathring{S}(f_1)$ and SU(m/n), and $|\mathring{Y}^{\nu_2}_{m_2}(\omega_2), \mathring{W}_2\rangle$ the IRB of $\mathring{S}(f_2)$ and SU(p/q). Equation (4b) is the SU($m+p/n+q$) ⊃ SU(m/n)⊗SU(p/q) IRB. Therefore using the ORC one can easily construct such a basis for arbitrary m, n, p and q.

5. The SU(m/n) ⊃ SU(m) × SU(n) IRB

The graded unitary group SU(m/n) also contains the ordinary unitary group SU(m) × SU(n) as its subgroup. With equations (3d) and (4b), we know that the SU(m/n) ⊃ SU(m) × SU(n) IRB can be constructed in the following way:

$$|\mathring{Y}^{\nu}_{m,\beta\nu_1W_1\nu_2W_2}{}^{[\nu]}\rangle = \sum_{m_1m_2\omega} C^{[\nu]\beta,m}_{\nu_1m_1,\tilde{\nu}_2\tilde{m}_2,\omega}|Y^{\nu_1}_{m_1}(\omega_1), W_1\rangle|Y^{\nu_2}_{m_2}(\omega_2), W_2\rangle \quad (5)$$

where the meaning of $|Y^{\nu_i}_{m_i}(\omega_i), w_i\rangle$ is the same as in equation (4a), while $|\mathring{Y}^{\nu}_{m,\beta\nu_1W_1\nu_2W_2}{}^{[\nu]}\rangle$ is the Yamanouchi basis $[\nu]m$ of $\mathring{S}(f)$ as well as the SU(m/n) ⊃ SU(m) × SU(n) basis $[\nu]\beta\nu_1W_1\nu_2W_2$ of the graded unitary group SU(m/n).

6. The CGC for the special Gel'fand basis of SU(m/n)

(a) The ORC can be written as an overlap between the coupled and uncoupled basis vectors (Chen *et al* 1978a)

$$C^{[\nu],\ \ m}_{\nu_1m_1,\ \nu_2m_2,\omega} = \langle Y^{\nu}_{m}|Y^{\nu_1}_{m_1}(\omega_1)Y^{\nu_2}_{m_2}(\omega_2)\rangle. \quad (6)$$

Since the Yamanouchi basis of the permutation group S(f) is just the special Gel'fand basis of SU(m), $m \geq f$ (Moshinsky 1966), the Young tableaux in equation (6) can be regarded as the Weyl tableaux, and the overlap thus becomes a CGC of SU(m). Therefore equation (6a) shows that the ORC of S(f) is the CGC for the special Gel'fand basis of SU(m) with $m \geq f$ (Chen *et al* 1978a).

(b) With the same reasoning, it can be shown that the ORC with appropriate modifications in the signs gives the CGC for the special Gel'fand basis of SU(m/n). As an example, table 1 shows the ORC of S(3) and the CGC of SU(m), while table 2 shows the CGC of the graded unitary group SU(m/n), where the state indices A, B and C can be either bosonic or fermionic, and $[BC]$ and $[A\,^{B}_{C}]$ are sign functions defined by Jarvis and Green (1979): $[BC] = (-1)^{(B+C)}$, $[A\,^{B}_{C}] = (-1)^{(A)(B+C)}$ and $(A) = 1\ (-1)$ for bosonic (fermionic) is the grade of state A.

The CGC problem for the general Gel'fand basis of SU(m/n) is being investigated.

7. The SU($mp+nq/mq+np$) ⊃ SU(m/n) × SU(p/q) ISF

(a) Let

$$\left|\begin{matrix}[\sigma]\\ \sigma'm'_1\sigma''m''_1\end{matrix}\right\rangle, \quad \left|\begin{matrix}[\mu]\\ \mu'm'_2\mu''m''_2\end{matrix}\right\rangle, \quad \left|\begin{matrix}[\nu]\\ \nu'm'\nu''m''\end{matrix}\right\rangle \quad (7)$$

Table 1. The ORC of S(3) and CGC of SU(m). For the ORC use the bottom and right headings, for the CGC of SU(m) use the top and left headings.

	$\boxed{a}\boxed{b}\ \boxed{c}$	$\boxed{a}\boxed{c}\ \boxed{b}$	$\boxed{b}\boxed{c}\ \boxed{a}$	
$\boxed{a}\boxed{b}\boxed{c}$	$(1/3)^{1/2}$	$(1/3)^{1/2}$	$(1/3)^{1/2}$	$\boxed{1}\boxed{2}\boxed{3}$
$\begin{smallmatrix}a&b\\c&\end{smallmatrix}$	$(4/6)^{1/2}$	$-(1/6)^{1/2}$	$-(1/6)^{1/2}$	$\begin{smallmatrix}1&2\\3&\end{smallmatrix}$
$\begin{smallmatrix}a&c\\b&\end{smallmatrix}$	0	$(1/2)^{1/2}$	$-(1/2)^{1/2}$	$\begin{smallmatrix}1&3\\2&\end{smallmatrix}$
	$\boxed{1}\boxed{2}\ \boxed{3}$	$\boxed{1}\boxed{3}\ \boxed{2}$	$\boxed{2}\boxed{3}\ \boxed{1}$	

Table 2. The CGC of SU(m/n).

	$\boxed{A}\boxed{B}\ \boxed{C}$	$\boxed{A}\boxed{C}\ \boxed{B}$	$\boxed{B}\boxed{C}\ \boxed{A}$
$\boxed{A}\boxed{B}\boxed{C}$	$(1/3)^{1/2}$	$(1/3)^{1/2}$	$(1/3)^{1/2}[A^{B}_{C}]$
$\begin{smallmatrix}A&B\\C&\end{smallmatrix}$	$(4/6)^{1/2}$	$-(1/6)^{1/2}[BC]$	$-(1/6)^{1/2}[A^{B}_{C}]$
$\begin{smallmatrix}A&C\\B&\end{smallmatrix}$	0	$(1/2)^{1/2}[BC]$	$-(1/2)^{1/2}[A^{B}_{C}]$

be the IRB of $S(f) \supset S(f_1) \times S(f_2)$ in the spaces of x, ξ and $q = (x, \xi)$ respectively. For simplicity, here and in the following, we drop all the multiplicity labels. The $S(f) \supset S(f_1) \times S(f_2)$ ISF (or the inner-product ISF) are defined as the expansion coefficients in the equation

$$\left|\begin{matrix}[\nu]\\ \nu' m' \nu'' m''\end{matrix}\right\rangle = \sum_{\sigma'\sigma''\mu'\mu''} C^{[\nu]\nu'\nu''}_{[\sigma]\sigma'\sigma'',[\mu]\mu'\mu''} \left[\left|\begin{matrix}[\sigma]\\ \sigma'\sigma''\end{matrix}\right\rangle \left|\begin{matrix}[\mu]\\ \mu'\mu''\end{matrix}\right\rangle\right]^{[\nu'][\nu'']}_{m'm''} \tag{8}$$

where the large square bracket indicates the coupling of $[\sigma'] \times [\mu'] \to [\nu']m'$ and $[\sigma''] \times [\mu''] \to [\nu'']m''$ in terms of the CGC of $S(f_1)$ and $S(f_2)$ respectively. Let

$$\left|\begin{matrix}[\nu']\\ \sigma' W_1' \mu' W_2'\end{matrix}\right\rangle \qquad \left|\begin{matrix}\nu''\\ \sigma'' W_1'' \mu'' W_2''\end{matrix}\right\rangle \qquad \left|\begin{matrix}[\nu]\\ \sigma W_1 \mu W_2\end{matrix}\right\rangle \tag{9}$$

be the IRB of $SU(mn) \supset SU(m) \times SU(n)$ in the q-space for the particles $(1, 2, \ldots, f_1)$, $(f_1+1, \ldots, f)$ and $(1, 2, \ldots, f)$ respectively. The $SU(mn) \supset SU(m) \times SU(n)$ f_2-body ISF are defined by the expansion coefficients in

$$\left|\begin{matrix}[\nu]\\ \sigma W_1 \mu W_2\end{matrix}\right\rangle = \sum_{\sigma'\mu'\sigma''\mu''} C^{[\nu]\sigma\mu}_{[\nu']\sigma'\mu'[\nu'']\sigma''\mu''} \left[\left|\begin{matrix}[\nu']\\ \sigma'\mu'\end{matrix}\right\rangle \left|\begin{matrix}[\nu'']\\ \sigma''\mu''\end{matrix}\right\rangle\right]^{[\sigma][\mu]}_{W_1 W_2} \tag{10a}$$

with the large square bracket indicating the coupling of $[\sigma'] \times [\sigma''] \to [\sigma]W_1$ and $[\mu'] \times [\mu''] \to [\mu]W_2$ in terms of the CGC of SU(m) and SU(n) respectively. The $SU(mn) \supset SU(m) \times SU(n)$ ISF has been identified with the $S(f) \supset S(f_1) \times S(f_2)$ ISF (Chen 1981), i.e.

$$C^{[\nu]\sigma\mu}_{[\nu']\sigma'\mu',[\nu'']\sigma''\mu''} = C^{[\nu]\nu'\nu''}_{[\sigma]\sigma'\sigma'',[\mu]\mu'\mu''}. \tag{11}$$

Systematic tables of the $SU(mn)\supset SU(m)\times SU(n)$ one-, two- and three-body ISF for up to the six-particle system are available (Chen *et al* 1982b, Shi *et al* 1982a,b).

(b) It can be shown that the $S(f)\supset S(f_1)\times S(f_2)$ ISF are also the $SU(mp+nq/mq+np)\supset SU(m/n)\times SU(p/q)$ ISF; therefore we have

$$\left|\begin{matrix}[\nu]\\ \sigma\mathring{W}_1\mu\mathring{W}_2\end{matrix}\right\rangle=\sum_{\sigma'\sigma''\mu'\mu''} C^{[\nu]\nu'\nu''}_{[\sigma]\sigma'\sigma''[\mu]\mu'\mu''}\left[\left|\begin{matrix}[\nu']\\ \sigma'\mu'\end{matrix}\right\rangle\left|\begin{matrix}[\nu'']\\ \sigma''\mu''\end{matrix}\right\rangle\right]^{[\sigma][\mu]}_{\mathring{W}_1\mathring{W}_2} \tag{10b}$$

where

$$\left|\begin{matrix}[\nu']\\ \sigma'\mu'\end{matrix}\right\rangle,\quad \left|\begin{matrix}[\nu'']\\ \sigma''\mu''\end{matrix}\right\rangle\quad\text{and}\quad \left|\begin{matrix}[\nu]\\ \sigma\mathring{W}_1\mu\mathring{W}_2\end{matrix}\right\rangle$$

are the $SU(mp+nq/mq+np)\supset SU(m/n)\times SU(p/q)$ IRB in the $q=(x,\xi)$ space for the particles $(1,2,\ldots f_1)$, $(f_1+1,\ldots f)$ and $(1,2,\ldots f)$ respectively, and the large square bracket denotes the couplings of $[\sigma']\times[\sigma'']\rightarrow[\sigma]\mathring{W}_1$ and $[\mu']\times[\mu'']\rightarrow[\mu]\mathring{W}_2$ in terms of the CGC of $SU(m/n)$ and $SU(p/q)$.

8. The $SU(m+p/n+q)\supset SU(m/n)\otimes SU(p/q)$ ISF

(a) Let f_1, f_2, f_3 and f_4 be four integers, let

$$\begin{aligned} &f_{12}=f_1+f_2 \qquad f_{34}=f_3+f_4 \qquad f_{13}=f_1+f_3\\ &f_{24}=f_2+f_4 \qquad f=f_{12}+f_{34}=f_{13}+f_{24}\end{aligned} \tag{12}$$

and let the three states in equation (7) be reinterpreted as the IRB for the group chains $S(f_{13})\supset S(f_1)\times S(f_3)$, $S(f_{24})\supset S(f_2)\times S(f_4)$ and $S(f)\supset S(f_{12})\times S(f_{34})$, respectively. Under this provision, equation (8) is now understood as a definition for the $S(f)\supset S(f_{12})\times S(f_{34})$ outer-product ISF, with the large square bracket indicating that the bases are to be combined into the IRB of $S(f_{12})$ and $S(f_{34})$ by the ORC of $[\sigma']\otimes[\mu']\rightarrow[\nu']m'$ and $[\sigma'']\otimes[\mu'']\rightarrow[\nu'']m''$, respectively.

On the other hand, the three states in equation (9) are now regarded as the IRB of $SU(m+n)\supset SU(m)\otimes SU(n)$ for the particles $(1,2,\ldots,f_{12})$, $(f_{12}+1,\ldots,f)$ and $(1,2,\ldots,f)$, respectively, while equation (10a) serves as the definition for the $SU(m+n)\supset SU(m)\otimes SU(n)$ f_{34}-body ISF. It can be proved that the $SU(m+n)\supset SU(m)\otimes SU(n)$ f_{34}-body ISF are equal to the $S(f)\supset S(f_{12})\times S(f_{34})$ outer-product ISF; therefore equation (11) still holds under this new interpretation.

The $SU(m+n)\supset SU(m)\otimes SU(n)$ one-body ISF for up to the six-particle system have been calculated (Chen and Gao 1982b).

(b) Analogously, it can be shown that the $S(f)\supset S(f_{12})\times S(f_{34})$ outer-product ISF are also the $SU(m+p/n+q)\supset SU(m/n)\otimes SU(p/q)$ ISF. Therefore we have

$$\left|\begin{matrix}[\nu]\\ \sigma\mathring{W}_1\mu\mathring{W}_2\end{matrix}\right\rangle=\sum_{\sigma'\sigma''\mu'\mu''} \bar{C}^{[\nu]\nu'\nu''}_{[\sigma]\sigma'\sigma'',[\mu]\mu'\mu''}\left[\left|\begin{matrix}[\nu']\\ \sigma'\mu'\end{matrix}\right\rangle\left|\begin{matrix}[\nu'']\\ \sigma''\mu''\end{matrix}\right\rangle\right]^{[\sigma][\mu]}_{\mathring{W}_1\mathring{W}_2} \tag{13}$$

where

$$\left|\begin{matrix}[\nu']\\ \sigma'\mu'\end{matrix}\right\rangle,\quad \left|\begin{matrix}[\nu'']\\ \sigma''\mu''\end{matrix}\right\rangle\quad\text{and}\quad \left|\begin{matrix}[\nu]\\ \sigma\mathring{W}_1\mu\mathring{W}_2\end{matrix}\right\rangle$$

are the $SU(m/n)$, $SU(p/q)$ and $SU(m+p/n+q)\supset SU(m/n)\otimes SU(p/q)$ IRB for the particles $(1,2,\ldots f_{12})$, $(f_{12}+1,\ldots f)$ and $(1,2,\ldots f)$, respectively. The expansion coefficients are the $S(f)\supset S(f_{12})\times S(f_{34})$ outer-product ISF, where a bar is used to emphasise that it is the outer-product rather than the inner-product ISF of $S(f)$.

9. The SU(m/n) ⊃ SU(m) × SU(n) ISF

Finally let the three states in equation (9) be considered as the SU(m/n) ⊃ SU(m) × SU(n) IRB for the particles $(1, 2, \ldots, f_{12})$, $(f_{12}+1, \ldots, f)$ and $(1, 2, \ldots, f)$, respectively. On the basis of equation (3d), it is readily seen that the SU(m/n) ⊃ SU(m) × SU(n) ISF are also related to the $S(f) \supset S(f_{12}) \times S(f_{34})$ outer-product ISF in a simple fashion, namely

$$\left|\begin{matrix}[\nu] \\ \sigma W_1 \mu W_2\end{matrix}\right\rangle = \sum_{\sigma'\sigma''\mu'\mu''} C^{[\nu]\nu'\nu''}_{[\sigma]\sigma'\sigma'',[\tilde{\mu}]\tilde{\mu}'\tilde{\mu}''} \left[\left|\begin{matrix}[\nu'] \\ \sigma'\mu'\end{matrix}\right\rangle \left|\begin{matrix}[\nu'] \\ \sigma''\mu''\end{matrix}\right\rangle\right]^{[\sigma][\mu]}_{W_1 W_2} \tag{14}$$

where the tilde denotes the conjugated partition.

10. Conclusions

The above outlines suffice to show that through a straightforward extension of the known results for the ordinary unitary groups, the seemingly formidable problem of the CGC and ISF of the graded unitary groups can be solved quite easily. Almost all these coefficients are already available, or may be calculated from existing programs. The reduction rules or branching rules given by Dondi and Jarvis (1981) are natural consequences of the formulae here involving the various kinds of coefficients.

References

Balantekin A B and Bars I 1981a *J. Math. Phys.* **22** 1149

—— 1981b *J. Math. Phys.* **22** 1810

Balantekin A B, Bars I and Iachello F 1981 *Nucl. Phys.* A **370** 284

Chen J Q 1981 *J. Math. Phys.* **22** 1

Chen J Q, Chen X G and Gao M J 1982a to be published

Chen J Q and Gao M J 1981 *Reduction Coefficients of Permutation Groups* (Beijing: Science)

—— 1982a *J. Math. Phys.* **23** 928

—— 1982b to be published

Chen J Q, Vallieres M Feng D H and Shi Y J 1982b to be published

Chen J Q, Wang F and Gao M J 1977a *Acta Phys. Sinica* **26** 307 (trans. 1981 *Chinese Phys.* **1** 533)

—— 1977b *Acta Phys. Sinica* **26** 427 (trans. 1981 *Chinese Phys.* **1** 542)

—— 1978a *Acta Phys. Sinica* **27** 31

—— 1978b *Acta Phys. Sinica* **27** 203

Dondi P H and Jarvis P D 1979 *Phys. Lett.* **84B** 75

—— 1981 *J. Phys. A: Math. Gen.* **14** 547

Jarvis P D and Green H S 1979 *J. Math. Phys.* **20** 2115

Moshinsky M 1966 *J. Math. Phys.* **7** 691

Ne'eman Y 1979 *Phys. Lett.* **81B** 190

Partensky A 1972a *J. Math. Phys.* **13** 621

—— 1972b *J. Math. Phys.* **13** 1503

Shi Y J *et al* 1982a to be published

—— 1982b to be published

Shindler S and Mirman R 1977 *J. Math. Phys.* **18** 1697

Sun X Z and Xan Q Z 1981 *Scientia Sinica* **24** 914

Taylor J G 1979 *Phys. Rev. Lett.* **43** 824

Xan Q Z *et al* 1981 *Phys. Energ. Fortis Phys. Nucl.* **5** 546